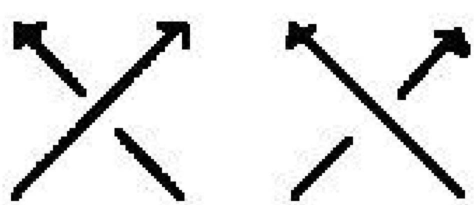
Polynomial Invariants of Knots

Why would we need a polynomial to identify a knot? Well, if we were to use simply a number, such as the bridge number or the crossing number, many knots may have the same number. Also, these are very difficult to calculate, as it is a minimum over all the possible configurations of the knot, and since there are infinitely many configurations, this requires some clever arguments to give a proof of the minimum. The polynomials can be calculated by a simple, albeit long, algorithm. A few terms need to be defined in order for the polynomials to be possible to calculate. The first such term is orientation.

Orientation and Writhe

For any knot, we can define an **orientation** or direction of the knot. To do this, we simply put an arrow somewhere on the knot and state that that arrow specifies the direction. This is useful in defining what the writhe is.

The **writhe** of a knot is defined to be the sum of all the crossings of the knot. We give each crossing a number: either -1 or 1. Here is how we can tell whether a knot gets a 1 or a -1 number:



+1 −1

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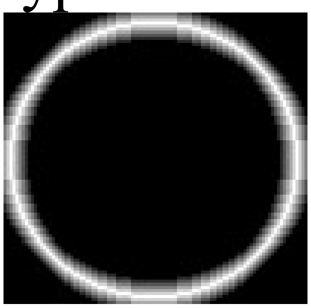
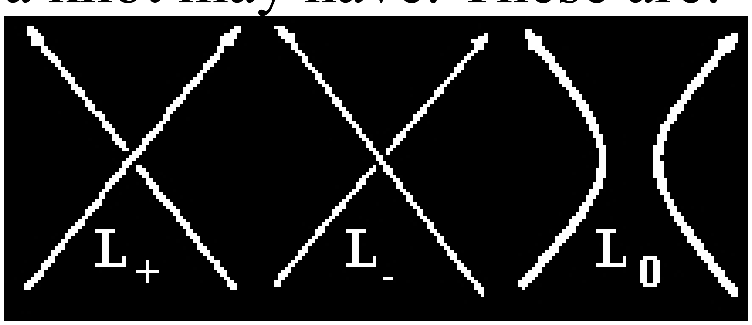
The Unknot and Skein Relations

One type of knot is called an unknot. An **unknot** is simply a circle. It is the simplest type of knot. An unknot looks like this:

.

There are three types of **skein relations** that a knot may have. These are: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Introduction to Polynomials

Now we know everything we need to know in order to be able to define the three most important types of polynomial invariants of knots. Actually, these are not polynomials in the strict sense of the word. Rather, they are Laurent polynomials, which means that they may have some negative exponents in their expansion. The first polynomial is the Alexander polynomial.

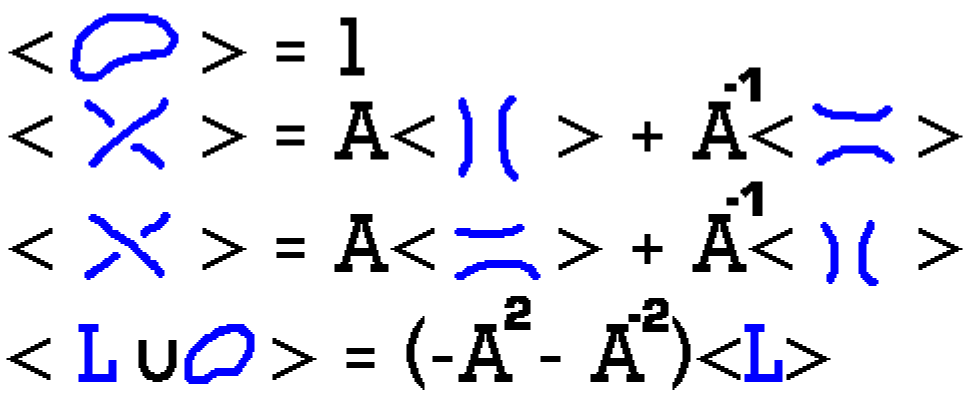
Alexander Polynomial

The **Alexander polynomial** of a knot was the first polynomial invariant discovered. It was discovered in 1928 by J. W. Alexander, and until the 1980's, it was the only polynomial invariant known. Alexander used the determinant of a matrix to calculate the Alexander polynomial of a knot. But this can also be done using the skein relations. There are three rules that are used when calculating the Alexander polynomial of a knot:

1) $\Delta(unknot) = 1$
2) $\Delta(2\ unknots) = 0$
3) $\Delta(L_+) - \Delta(L_-) + \left(t^{1/2} - t^{-1/2}\right)\Delta(L_0) = 0$

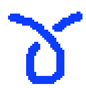
Bracket Polynomials

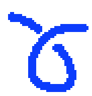
The next type of polynomial is the **bracket polynomial**, which is very similar to the **Kauffman polynomial** and the **Jones polynomial**. The bracket polynomial must be calculated first in order to get to the Kauffman polynomial and the Jones polynomial. Again, there are several rules for calculating the bracket polynomial of a knot:



Why the Bracket Polynomial is not Invariant

This polynomial, however, is not an invariant. Under the Reidemeister move R1, we get:

$\langle$  $\rangle = -A^{3} \langle$  $\rangle$

$\langle$  $\rangle = -A^{-3} \langle$  $\rangle$

So now we will have a bracket polynomial for our knot.

Kauffman and Jones Polynomials

We can easily transform our bracket polynomial into a Kauffman polynomial by means of the following equation: $F[L] = (-A)^{-3\,w(D)} \langle D \rangle$.

The Jones polynomial is now extremely similar to the Kauffman polynomial. Because the exponents in the Kauffman polynomial become very large often, we can decrease them and thus reach the Jones polynomial by means of the transformation: $V_L(t) = F[L]\left(t^{-1/4}\right)$.

HOMFLY Polynomial

The **HOMFLY polynomial** is a generalization of the Alexander and Jones polynomials. Instead of being a polynomial in one variable as the other two are, it is a polynomial in two variables. It was discovered in 1985 by J. Hoste, A. Ocneanu, K. C. Millett, P. J. Freyd, W. B. R. Lickorish, and D. N. Yetter. For this polynomial, there are only two rules:

1) $P(unknot) = 1$
2) $l\,P(L_+) + l^{-1}\,P(L_-) + m\,P(L_0) = 0$