Properties of the Pseudosphere

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1 Euclidean, Hyperbolic, and Elliptic Geometries

Euclid began his study of geometry with five axioms. They are

1. Exactly one line may drawn from any point to any other point.
2. Any segment can be extended to a line.
3. Exactly one circle may be formed with a given center and radius.
4. All right angles are congruent.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

The fifth axiom, the so–called "parallel postulate," is much clumsier than the first four, so many people thought it should be possible to prove it from the first four axioms. There is a simpler way of stating the parallel postulate:

Playfair's Axiom *For every line l and point $P \notin l$, there is a unique line l' that passes through P and is parallel to l.*

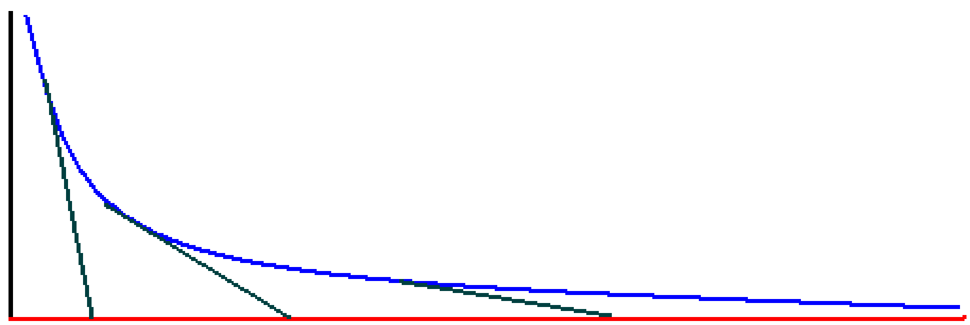
Ultimately, all the mathematicians who tried to prove the parallel postulate from the first four axioms were doomed to failure since it is impossible: the parallel postulate is independent of the first four axioms. Naturally, a few mathematicians, such as Karl Friedrich Gauss, János Bolyai, Nicolai Lobachevsky, Eugenio Beltrami, and Bernhard Riemann were happy to investigate the properties of geometries that assumed that the fifth postulate was in fact false. These geometries are called non–Euclidean geometries. There are two non–Euclidean geometries, each of which have their own fifth axiom.

Elliptic Geometry For every line l and point $P \notin l$, there are no lines that pass through P and are parallel to l.

Hyperbolic Geometry For every line l and point $P \notin l$, there are at least two lines that pass through P and are parallel to l.

The pseudosphere is a model for hyperbolic geometry.

2 The Tractrix



A tractrix is a curve with the property that the segment of the tangent line between the point of tangency and a fixed line is constant. A tractrix is the curve that, when revolved around the fixed line, forms a pseudosphere.

3 Curvature

Let $\sigma(u, v)$ be a surface of revolution parametrized as usual by

$$\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u)).$$

We know that

$$K = -\frac{\ddot{f}}{f}.$$

If K is to be constant and negative, then WLOG $K \equiv -1$. This gives us $f - \ddot{f} = 0$. Solving this differential equation gives

$$f(u) = ae^u + be^{-u}.$$

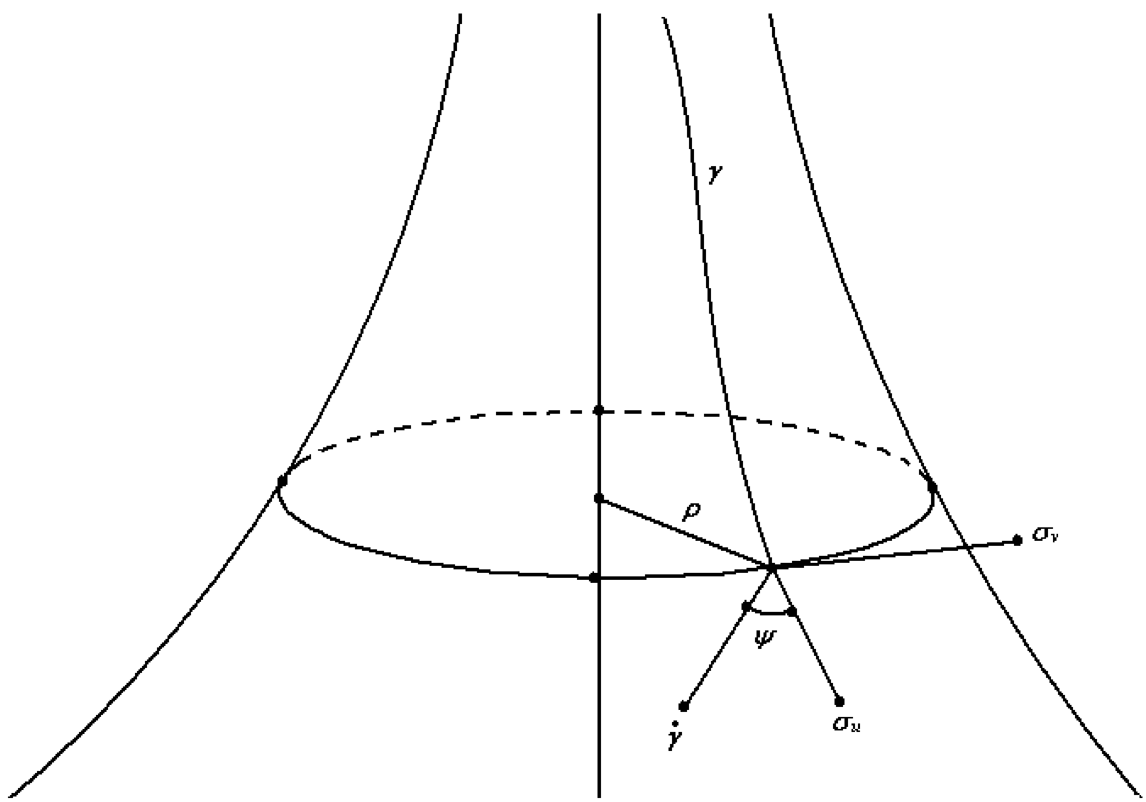
The case in which $a = 1$ and $b = 0$ has the added benefit of being solvable in terms of elementary functions, so we shall consider only this case. Since in the computation we performed earlier to derive the formula for K requires that $\dot{f}^2 + \dot{g}^2 = 1$, we get

$$g(u) = \int \sqrt{1 - e^{2u}}\, du = \sqrt{1 - e^{2u}} - \cosh^{-1}(e^{-u}) + C.$$

We can translate the pseudosphere to make $C = 0$ WLOG. If we now replace some variables, we find that the xz–cross section of a pseudosphere has the equation

$$z = \sqrt{1 - x^2} - \cosh^{-1} \frac{1}{x}.$$

4 Geodesics



Recall that Clairaut's Theorem states that if $\rho \sin\psi$ is constant along a curve γ so that no part of γ is part of a parallel, then γ is a geodesic. We shall apply this theorem to find geodesics on a pseudosphere. We use the parametrisation

$$\sigma(u,v) = \left(\frac{1}{u}\cos v, \frac{1}{u}\sin v, \sqrt{1-\frac{1}{u^2}} - \cosh^{-1} u\right),$$

which has FFF

$$\frac{1}{u^2}du^2 + \frac{1}{u^2}dv^2$$

with $u > 1$. Therefore, suppose that $\gamma(t) = \sigma(u(t), v(t))$ is a unit–speed geodesic on the pseudosphere. From the FFF, we get $\dot{u}^2 + \dot{v}^2 = u^2$, and so by Clairaut's Theorem,

$$\frac{1}{u}\sin\psi = \frac{1}{u^2}\dot{v} = \Xi$$

for some constant Ξ. (I feel that if Andrew Pressley can take the liberty to call his constants Ω, then I can call mine Ξ.) Therefore $\dot{v} = \Xi u^2$. If we take $\Xi = 0$, then we get a tractrix. If $\Xi \neq 0$, then

$$\dot{u} = \pm u\sqrt{1-\Xi^2 u^2}.$$

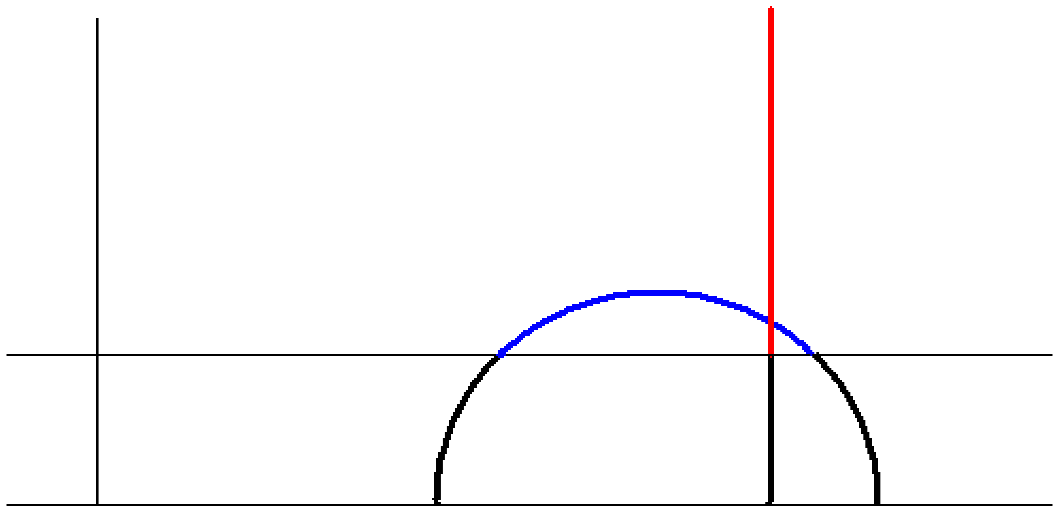
Then

$$\frac{\dot{v}}{\dot{u}} = \pm\frac{\Xi u}{\sqrt{1-\Xi^2u^2}},$$

$$(v - v_0) = \mp\frac{1}{\Xi}\sqrt{1-\Xi^2u^2},$$

$$u^2 + (v-v_0)^2 = \frac{1}{\Xi^2}$$

for some constant v_0. The red line in the vu–plane maps to a meridian (or a tractrix in this case), and the blue arc maps to a geodesic that is not a meridian. (Recall that it is necessary to have $u > 1$.)



The problem is that these geodesics can't be continued indefinitely: they have to end when they reach the rim of the pseudosphere. In this sense, the psedosphere isn't read the hyperbolic geometry equivalent of the plane. Also, the pseudosphere is not simply connected since not every rectifiable closed curve drawn on the pseudosphere can be deformed along the pseudosphere to a point. In this sense, a pseudosphere is more like a cylinder than a plane. The Poincaré Disc is a model of the hyperbolic plane that is closer to being analogous to the plane.